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A Variational Approach for Designing Infinite Impulse Response Filters with Time-Varying Parameters

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***Abstract*—Filter design with short transient state is a problem encountered in many fields of circuits, systems and signal processing. In this paper a novel low-pass filter design technique with time-varying parameters is introduced in order to minimize the rise-time parameter. Through the use of calculus of variations a method is developed to obtain the optimal closed-form expression for adjusting the parameters. In this context, two cases are addressed. The ideal case in which infinite bandwidth is required and a solution of finite bandwidth. The latter is obtained by means of a proper constraint formulation in the frequency domain. The proposed filter achieves the shortest rise-time and allows better preservation of the edge shape in comparison with other existing filtering methods. The analysis, synthesis and performance of the proposed system is discussed and illustrated with the aid of simulations.**

***Index Terms*—Filter design, rise-time, time-varying parameters, calculus of variations.**

I. Introduction

LINEAR Time Variant filters (LTV) are commonly applied to adjust time and frequency specifications simultaneously. Frequency specifications are usually considered to design low-pass, band-pass, high-pass or band-stop frequency responses, while time specifications comprise rise-time and overshooting parameters.

The concept is first addressed to deal with processing time and correctness trade-off [1]. The dynamic parameters of systems are varied in time to reduce the transient behavior as much as possible. In this regard, the pioneering work of Kaszynski in [2] addresses the design of low-pass filters with time-varying parameters. This approach significantly reduces the transient period in comparison with filters of constant parameters.

The shortening of the transient behavior is directly related to a reduction of rise-time. Through the use of LTV filters, several applications are reported to be used on a variety of fields of circuits, systems and signal processing such as: dynamic weighing systems [3], software defined radio (SDR) [4], sensors for biomedical applications [5], channel equalizers [6], inertial sensors [7] and radar processing [8], for instance.

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A. Prior Work

A variety of designs have been considered to reduce the rise-time parameter when rectangular pulses are received. Solutions reported to date either modify parameters on the S plane or by means of closed-form expressions in the time domain. These solutions are based on constant and time-varying linear filters.

1) Filters of Constant Parameters: Several configurations are based on the displacement of poles over particular curves in the S plane with regard to the Butterworth design, in order to obtain a reduction of rise-time. For instance, some designs use the parabola, the ellipse or the catenary curves [9].

Other solutions modify the expression of a given design by a parameter. In this respect, a modification of the Bessel filter is considered in [10] and the rise-time is reduced through the increase of a given parameter. However, the cutoff frequency is also shifted, which represents a trade-off for this type of solution. Moreover, a method for the synthesis of the wide-band amplifier transfer function can be developed by using the direct performance parameter in the time domain, known as delay to rise-time ratio [11].

On the other hand, the eigenfilter method represents an approach in the least squares sense [12]. This design is implemented with time and frequency constraints in order to minimize the rise-time and overshooting of the step response simultaneously. In a similar way, linear programming has been applied to filter design. In this approach the ease of implementation is remarkable and the convergence to a unique-solution is generally guaranteed in a frame of filters of constant parameters [13].

Furthermore, by means of several analytic definitions of the rise-time parameter, closed-form expressions can be obtained for the filter coefficients. These coefficients are obtained by looking for the minimum value of the rise-time. A first approach for defining the rise-time analytically, given in [14], is based on the standard deviation of the impulse response. A second approach, described in [15], is obtained under the constraint of a given noise bandwidth. Finally, after considering constraints in the bandwidth, optimal expressions of the filter coefficients are provided in [16].

2) Filters of Time-Varying Parameters: In case that a constant cutoff frequency is specified in advance, the indeterminacy principle establishes a lower bound for the rise-time. In order to overcome this issue, the variation of filter parameters in time is implemented. For instance, the characteristic frequency function of second-order filters is time-varied to achieve a reduction of rise-time at the filter output. This is implemented by linear time-varying (LTV) filters.

In the analog domain, the concept of LTV filters is incorporated into the design of lowpass filters by the Chebyshev [2] and Bessel [17], [18] approximations. Additionally, Notch [19] and phase compensated systems for Butterworth [20], Chebyshev [21] and Elliptic [22] filters are also implemented.

In the discrete-time domain, LTV filters are also implemented by FIR [23] and IIR [24] systems. Although, reports are focused on Notch filtering as described in [5], [25]. By means of LTV filters the characteristic function, the damping factor and the Q-parameter of second-order systems are modified, in a timely manner. The variation of filter parameters is normally adjusted by means of linear and exponential functions as well as curves derived from second order systems and Bézier curves [1], [25]. In addition, other reports establish nonzero initial conditions to further reduce the transient behavior [26].

B. Motivation

Based on reported solutions to reduce the rise-time parameter, filters with time-varying coefficients offer the best approach. In this concern, a variety of solutions modify the characteristic frequency as well as the damping factor of second-order lowpass IIR filters with a variety of rules. However, an optimal solution to describe the variations of the characteristic frequency function in time has not been considered. Additionally, an optimal solution in case of filters of arbitrary order is not available. The motivation of this work is the development of optimal expressions for the characteristic function of lowpass IIR filters of arbitrary order to reduce, as much as possible, the duration of the transient response.

C. Contribution

The contribution of this paper is the derivation of an optimal closed-form expression for the characteristic frequency function of lowpass IIR filters of arbitrary order, in order to minimize the rise-time. As stated above, previous proposals for the design of LTV filters were non-optimal. The optimal expression is obtained through the use of the mathematical tool of calculus of variations [27], that to the best of our knowledge, had not been previously applied to this problem. The current work is inspired by the work presented in [17] and further analyzed in [22], [28], [29]. Although the proposed solution is obtained for first-and second-order systems, the generalization to higher-order filters is straightforward and can be obtained by implementing a cascade structure comprised by first-and second-order filters. A practical approximation and an implementation of the proposed time-varying system are also discussed. The results show that the proposed approach has lower values of rise-time than the previously reported approaches to designing time-varying systems.

D. Organization

Section II presents the filter design concepts for ideal and practical cases. The characteristic frequency function of first- and second-order lowpass IIR filters is derived to optimally reduce the rise-time parameter. Section III discusses the frequency response in the time frequency plane and the stability of the proposed design in case of first and second order systems. Section IV presents the design procedure of higher-order filters. Software and hardware implementation followed by an illustrative example are also presented. Additionally, the extension to higher order lowpass IIR filters of arbitrary order is also presented. In Section V the evaluation of the proposed design is illustrated in comparison with other reported solutions followed by the concluding remarks and future work in Section VI.

II. Problem Formulation and Filter Design Concepts

A. Problem Formulation

Linear time-invariant systems are completely described in time and frequency domains by the impulse response $h(t)$ and the transfer function $H(s)$, respectively. In addition, given the step response, denoted by $g(t)$, of a linear time-invariant system, the rise-time is defined as the time interval T_R between $g(t) = 0.1$ and $g(t) = 0.9$. Moreover, it is known as the time required for the step response to increase from 10 to 90 percent of its final value [30]. However, to establish an analytic expression of T_R from this definition is not feasible.

An alternative rise-time definition can be derived taking into account the impulse response $h(t)$ as described in [14]. This definition is applied in particular fields as indicated in [31]. The quantity is directly related to the standard deviation of $h(t)$ and it is described by:

$$T_R = \sqrt{2\pi \left[\int_0^\infty t^2 h(t) \mathrm{d}t - T_D^2 \right]}, \tag{1}$$

where T_D is the time-delay and can be defined as the centroid of the area of the curve $h(t)$ [14], i.e.:

$$T_D = \int_0^\infty t \cdot h(t) \mathrm{d}t. \tag{2}$$

Graphically, this approach is illustrated in Fig.1. The value of T_R is related to the spreading of $h(t)$. The shorter the standard deviation of $h(t)$, the more pronounced the step response becomes.

Let us consider an analog second-order system, described by the following transfer function:

$$H_2(s) = \frac{k}{\dfrac{1}{\omega_0^2} s^2 + \dfrac{2\beta}{\omega_0} s + 1}, \tag{3}$$

where k, β and ω_0 represent the gain, the damping factor and the characteristic frequency, respectively. Expressions for varying the ω_0 and β parameters in time for the second-order system given in (3) are implemented by linear, exponential and second-order curves to reduce the transient behavior.

The parameters β and ω_0 are related to the step response. The larger the values of β and ω_0, the smaller the output oscillations and the rise-time become.

This dynamic system, described by an LTV filter, allows a reduction of rise-time. However, the optimal closed-form expressions have not been derived for second-order systems, first-order systems nor higher-order IIR filters, which in turn demands further analysis. The next subsection addresses a method for solving the above problem based on the definition given in (1) and the use of calculus of variations.

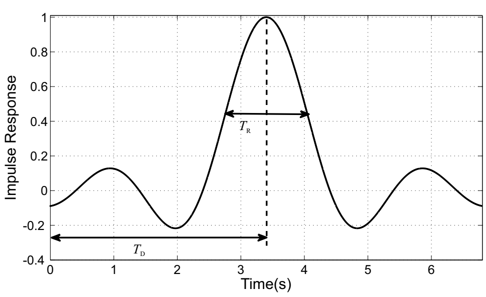

Fig. 1. Impulse response of a generic system. This figure depicts the function $sinc(t - t_0)$, in which $t_0 = 3.4$.

B. Optimal Solution: Ideal Case with Infinite Bandwidth

The aforementioned closed-form expression of rise-time in (1) can be considered as the functional of a variational problem of the form [27]:

$$I(\omega_0) = \int_0^{\infty} F(\omega_0, \omega_0', t)\mathrm{d}t, \tag{4}$$

where:

$$F(\omega_0, \omega_0', t) = t^2 \cdot h(\omega_0, t), \tag{5}$$

the quantities t, w_0 and $h(w_0, t)$ represent time, characteristic frequency and the impulse response in case of first-and second-order systems, respectively.

The integrand given in (4) is related to the definition of T_R in (1). The number $I(\omega_0)$, defined in (4), gives a measure of rise-time, since with lower $I(\omega_0)$ the rise-time assumes a smaller value.

The value of $T_D = 0$ is considered here, without loss of generality, to reduce complexity on the analytic solution of the variational problem. The solution to be obtained by this procedure is given in terms of coordinates centered at T_D in Fig. 1. The solution for $h(t)$ is then transformed to $h(t - T_D)$ to obtain the final solution.

The main idea is to find an optimal expression of $\omega_0(t)$, denoted by $\overline{w}_0(t)$, in order to obtain the optimal value of the T_R parameter. This solution must satisfy the Euler-Lagrange equation [27]:

$$\frac{\partial F\left(\omega_0,\, \omega_0',\, t\right)}{\partial \omega_0} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F\left(\omega_0,\, \omega_0',\, t\right)}{\partial \omega_0'}\right) = 0. \tag{6}$$

where $\frac{\partial}{\partial \omega}$ and $\frac{\mathrm{d}}{\mathrm{d}t}$ stand for partial and total derivatives, respectively.

Since the F function is independent of ω_0', the Euler-Lagrange equation is reduced to the first term on the left-hand side of (6). By writing out the derivatives and equating the terms to zero, the following condition arises:

$$\frac{\partial F(\omega_0, t)}{\partial \omega_0} = 0, \tag{7}$$

1) Filters of arbitrary order: Let $H(s)$ describe a rational transfer function of a time-invariant IIR filter of arbitrary order. In which the order of the numerator is zero, when Butterworth, Chebyshev and Elliptic design methods are applied [32]. Similar to the approach presented in [17], the time-varying IIR filter may be obtained after a cascade connection in terms of first-and second-order lowpass filters as:

$$H(s) = \prod_{i=1}^{N_1} H_{1i}(s) \cdot \prod_{i=1}^{N_2} H_{2i}(s), \tag{8}$$

where terms $H_{1i}(s)$ and $H_{2i}(s)$ are defined as:

$$\begin{aligned} H_{1i}(s) &= \frac{1}{\frac{1}{\omega_{1i}}s + 1}, \\ H_{2i}(s) &= \frac{1}{\frac{1}{\omega_{2i}^2}s^2 + \frac{2\beta_i}{\omega_{2i}}s + 1}, \end{aligned} \tag{9}$$

term $H_{1i}(s)$ stands for first-order systems, while second-order systems is determined by $H_{2i}(s)$.

Decomposition of the IIR filter in terms of systems of first- and second-order, allows to reduce the mathematical complexity of the problem. The next sections are devoted to describing the procedure to the characteristic frequency function in case of first and second-order systems.

2) First-Order Systems: Considering the transfer function of a first-order system as indicated by $H_{1i}(s)$ in (9), the impulse response is obtained by using the table of Laplace transforms from [33] as:

$$h_1(t) = \omega_0 e^{-\omega_0 t}, \tag{10}$$

where w_0 stands for the characteristic frequency.

Then, applying the condition in (7) the optimal expression for $\overline{w}_0(t)$ is obtained by:

$$\frac{\partial}{\partial \omega_0} t^2 \cdot h_1(\omega_0, t) = t^2\, e^{-\omega_0 t}\ (1 - t\omega_0) = 0. \tag{11}$$

This condition offers a solution for the characteristic frequency as:

$$\overline{\omega}_0(t) = \frac{1}{t}. \tag{12}$$

3) Second-Order Systems: Taking into account the term $H_{2i}(s)$ from (9), and by using the table of Laplace transforms in [33] the impulse response of second-order systems is obtained as:

$$h_2(t) = \frac{\omega_0}{\sqrt{1-\beta^2}} e^{-\beta\omega_0 t} \sin(w_0\sqrt{1-\beta^2}\, t). \tag{13}$$

Then, upon evaluating the condition in (7), and by using the following trigonometric identity:

$$A\cos(\theta+\alpha_1) + B\cos(\theta+\alpha_2) = R\cos(\theta+\phi), \tag{14}$$

where

$$\begin{aligned} R =& \sqrt{A^2+B^2+2AB\cos(\alpha_1-\alpha_2)}, \\ \phi =& tan^{-1}\frac{A\sin\alpha_1 + B\sin\alpha_2}{A\cos\alpha_1 + B\cos\alpha_2}. \end{aligned} \tag{15}$$

the following relation is derived:

$$\frac{\partial}{\partial\omega_0} t^2 \cdot h_2(\omega_0, t) = \frac{t^2 e^{-\beta\omega_0 t}}{\sqrt{1-\beta^2}} R_2 \cos(\theta+\phi_2), \tag{16}$$

where

$$\begin{aligned} R_2 =& \sqrt{\omega_0^2 t^2 - 2\,\beta\omega_0 t + 1}, \\ \theta =& w_0\sqrt{1-\beta^2}\, t \\ \phi_2 =& -\, tan^{-1}\frac{1-\beta t\omega_0}{t\omega_0\sqrt{1-\beta^2}}. \end{aligned} \tag{17}$$

Then, the condition $\frac{\partial}{\partial\omega_0} t^2 \cdot h_2(\omega_0, t) = 0$ is fulfilled whenever $R_{21} = 0$ or $\theta + \phi_{21} = \frac{(2p+1)\pi}{2}$, $p \in \mathbf{Z}$. Based on the first condition, two solutions are obtained after factorization given by:

$$\overline{\omega}_{0_{1,2}}(t) = \frac{1}{t}(\beta \pm \sqrt{(\beta-1)\ (\beta+1)}). \tag{18}$$

Additionally, based on the second condition the following expression is obtained:

$$t\omega_0\sqrt{1-\beta^2} - tan^{-1}\frac{1-\beta t\omega_0}{t\omega_0\sqrt{1-\beta^2}} = \frac{(2p+1)\pi}{2}, \tag{19}$$

which represents an implicit function of ω and must be solved by numerical methods to obtain the derivation of $\overline{\omega}_0(t)$.

4) Concluding remarks: Solutions of first-and second-order systems are described by the relations given in (12), (18) and (19). However, in all cases the characteristic frequency goes to infinity when t approaches zero. That is, the system cannot be implemented by any practical system in hardware. However, this problem can be addressed adding an energy restriction for the $\overline{\omega}_0(t)$ curve. This is described in the next Sub-Section.

C. Optimal Solution: Practical Case with Finite Bandwidth

The class of problems in which the required functional is expressed with integral restrictions is called isoperimetric [27]. The isoperimetric problem allows to establish conditions for obtaining bounded solutions. In this case, the constraint introduced in energy is described as:

$$E_{\omega_0} = \int_0^\infty \omega_0^2(t)\mathrm{d}t = C_1, \quad C_1 > 0, \tag{20}$$

where C_1 is a constant value. The condition (20) ensures a bounded solution of $\omega_0(t)$ for the variational problem in (4). In this case the new expression for the functional can be rewritten as follows:

$$F(\omega_0, \omega_0', t) = t^2 \cdot h(\omega_0, t) + \lambda_1\omega_0^2, \tag{21}$$

where λ_1 is the undetermined Lagrangian multiplier whose values must be determined. Thus, according to the Euler-Lagrange equation given in (6) with the functional described in (21), the following condition must be satisfied:

$$t^2 \cdot \frac{\partial h(\omega_0, t)}{\partial\omega_0} + \lambda_1\omega_0 = 0. \tag{22}$$

In this case the constant λ_1 in (22) is replaced by $2\lambda_1$ in (23) for the ease of representation.

1) First-Order System: Upon substituting the relation in (11) into (22) the following condition is established:

$$t^2\, e^{-t\omega_0}\ (1 - t\omega_0) + \lambda_1\omega_0 = 0, \tag{23}$$

in case of first-order systems.

2) Second-Order System: Similar to first-order systems, the following condition is obtained taking into account the relation in (16):

$$\frac{t^2 e^{-\beta\omega_0 t}}{\sqrt{1-\beta^2}} R_{21}\cos(\theta+\phi_{21}) + \lambda_1\omega_0 = 0, \tag{24}$$

where R_{21}, θ and ϕ_{21} are determined as indicated in (17).

3) Concluding Remarks: From (23) and (24) the derivation of a closed-form expression for $\overline{\omega}_0(t)$ is not affordable for both cases, first-and second-order systems. The solution must be determined by numerical methods. For each value of t a value of $\overline{\omega}_0(t)$ is determined whenever the left-hand side of (23) and (24) is equal to zero for first-and second-order systems, respectively.

Considering the solutions above for $\overline{\omega}_0(t)$ in (23) and (24), the implicit function theorem [34] guarantees the continuity of the real function $\omega_0(t)$ and thus $\omega_0^2(t)$. Moreover, given that $\int_0^\infty \omega_0^2(t)\mathrm{d}t = C_1$, as imposed in (20), then $\omega_0^2(t)$ will be a bounded function. According to this, the real function $\omega_0(t)$ is also bounded.

Although the proposed solutions in (23) and (24) for the characteristic frequency are obtained for finite bandwidth, it does not represent a stable filter. The next section is devoted to the analysis of stability, which in turn imposes an additional restriction on the characteristic frequency. In this case, the solution for the characteristic frequency function, to be devised on the next Section, determines a stable system of finite bandwidth.

III. Filter Analysis

Given the solutions derived for first-and second-order filters, the analysis in terms of stability and frequency response is described here.

A. Stability of the Proposed Method

The stability properties of the system with time-varying parameters are considered taking into account the discussion presented in [22], [28], [29]. In this regard, first-and second-order systems are modeled by scalar linear time-varying differential equations. In case of first-order systems the differential equation is given by:

$$y'(t) + \omega_0(t) y(t) = \omega_0(t) x(t), \tag{25}$$

taking into account the relation for $H(s)_{1i}$ in (9), where $y(t)$ and $x(t)$ represent, respectively, the output and the input of the filter. In case of second-order systems the differential equation is described by:

$$y''(t) + 2\beta\omega_0(t) y'(t) + \omega_0^2(t) y(t) = \omega_0^2(t) x(t). \tag{26}$$

Both equations in (25) and (26) may be rewritten through the use of matrix notation as:

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = F(t) \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} + G(t) z(t), \tag{27}$$

where $y_1(t) = y(t)$, $y_2(t) = y'(t)$ and $F(t)$ and $G(t)$ are matrices defined as follows for first-order systems:

$$F(t) = \begin{bmatrix} -\omega_0(t) & 0 \\ 0 & 0 \end{bmatrix} \quad G(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{28}$$

and second-order systems:

$$F(t) = \begin{bmatrix} 0 & 1 \\ -\omega_0^2(t) & -2\beta\omega_0(t) \end{bmatrix} \quad G(t) = \begin{bmatrix} 0 \\ \omega_0^2(t) \end{bmatrix}. \tag{29}$$

From [28], the proposed system has bounded-input bounded-output (BIBO) stability if the following conditions are met:

- The elements of $F(t)$ and $G(t)$ are bounded.
- The homogeneous response of (27) shows an exponential asymptotic stability.

Considering the first condition above, the real function $\omega_0(t)$ and thus $\omega_0^2(t)$ are bounded functions as discussed in Section II-C3. Thus, the first condition is fulfilled for first-and second-order systems by means of the relations in (23) and (24), respectively.

On the other hand, the exponential asymptotic stability can be addressed by the Poincaré-Lyapunov theorem [35] as follows. If the matrix $F(t)$ given by (29) is considered as:

$$F(t) = A + B(t), \tag{30}$$

then, the homogeneous response of (27) is exponential asymptotic stable if the two following conditions are satisfied:

1) A is a constant matrix with all eigenvalues having negative real part.
2) $B(t)$ is a continuous matrix with the property: $\lim_{t\to\infty} ||B(t)|| = 0$.

In order to analyze the above conditions, the elements of A are defined as follows:

$$a_{ij} = \alpha_{ij} \cdot \lim_{t\to\infty} \int_0^t ||f_{ij}(t)|| \mathrm{d}t, \tag{31}$$

where f_{ij} are the elements of $F(t)$ and α_{ij} are undetermined positive constants.

In addition, another constraint is introduced in the isoperimetric problem in order to satisfy that A is a constant matrix:

$$DC_{\omega_0} = \int_0^\infty \omega_0(t) \mathrm{d}t = C_2, \quad C_2 > 0, \tag{32}$$

where C_2 is the direct component of $\omega_0(t)$. The integral given by DC_{ω_0} determines the constant component of $\omega_0(t)$.

By means of the restriction in (32) and the Euler-Lagrange equation described in (6), a new condition for the variational problem must be satisfied in case of first-order systems:

$$t^2 \, e^{-t\omega_0} \, (1 - t\omega_0) + \lambda_1\omega_0 + \lambda_2 = 0, \tag{33}$$

and second-order systems as:

$$\frac{t^2 e^{-\beta\omega_0 t}}{\sqrt{1-\beta^2}} R_{21} \cos(\theta + \phi_{21}) + \lambda_1\omega_0 + \lambda_2 = 0, \tag{34}$$

where λ_2 is the undetermined Lagrangian multiplier whose values must be determined, R, θ and ϕ are given in (17).

1) First-Order Systems: Considering the first condition of the Poincaré-Lyapunov theorem, the matrix A is obtained as follows:

$$A = \begin{bmatrix} -\alpha_{11} C_2 & 0 \\ 0 & 0 \end{bmatrix}, \tag{35}$$

using the definition in (31) and the expression for $F(t)$ in (28). In this case, the eigenvalue is obtained by the relation:

$$\mu = -\alpha_{11} C_2, \tag{36}$$

which represents a negative value given that α_{11} and C_2 are positive constants.

Considering the second condition of the Poincaré-Lyapunov theorem, in order to guarantee that the equality in (30) holds, the matrix $B(t)$ is expressed as:

$$B(t) = \begin{bmatrix} -\omega_0(t) + \alpha_{11} C_1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{37}$$

The values of $B(t)$ tend to zero whenever the following condition is satisfied:

$$\lim_{t\to\infty} ||B(t)|| = \begin{bmatrix} -\omega_0(\infty) + \alpha_{11} C_1 & 0 \\ 0 & 0 \end{bmatrix} = 0, \tag{38}$$

which establishes the following condition for $\omega_0(\infty)$:

$$\omega_0(\infty) = \alpha_{11} C_1. \tag{39}$$

2) Second-Order Systems: Considering the first condition of the Poincaré-Lyapunov theorem, by writing out the expression given in (31), and through the use of the definition of $F(t)$ in (29), the matrix A is described as:

$$A = \begin{bmatrix} 0 & 1 \\ -\alpha_{21}C_1 & -2\alpha_{22}\beta C_2 \end{bmatrix}, \tag{40}$$

and the eigenvalues may be written down as:

$$\mu_{1,2} = 0.5\left(-2\alpha_{22}\beta C_2 \pm \sqrt{4\alpha_{22}^2\beta^2C_2^2 - 4\alpha_{21}C_1}\right). \tag{41}$$

Considering the expression obtained in (41), the first term will be a negative number given that β describes the damping factor and it is thus positive. The value of α_{22} is also defined to be a positive quantity.

On the other hand, the second term is an imaginary value since the argument of the square root procedure satisfies the condition $4\alpha_{22}^2\beta^2C_2^2 - 4\alpha_{21}C_1 < 0$ equivalent to :

$$C_1 > \frac{\alpha_{22}^2}{\alpha_{21}}\beta^2C_2^2. \tag{42}$$

Indeed, the energy associated to $\omega_0(t)$, described by the term C_1, is greater than the energy associated to the direct component of $\omega_0(t)$, described by the term C_2^2. Thus, $C_1 > C_2^2$ and considering also that $\frac{\alpha_{22}^2}{\alpha_{21}}\beta^2 < 1$, then the condition $C_1 > \frac{\alpha_{22}^2}{\alpha_{21}}\beta^2C_2^2$ is fulfilled when:

$$\beta < 1, \tag{43}$$

and

$$1 < \frac{\alpha_{22}^2}{\alpha_{21}} < \frac{1}{\beta^2}. \tag{44}$$

Hence, the eigenvalues of the matrix A have a negative real part.

Considering the second condition of the Poincaré-Lyapunov theorem, in order to guarantee that the equality in (30) holds, the matrix $B(t)$ is expressed as:

$$B(t) = \begin{bmatrix} 0 & 0 \\ -\omega_0^2(t) + \alpha_{21}C_1 & -2\beta\omega_0(t) + 2\alpha_{22}\beta C_2 \end{bmatrix}. \tag{45}$$

For large values of t the matrix $B(t)$ is transformed as:

$$\lim_{t\to\infty}||B(t)|| = \begin{bmatrix} 0 & 0 \\ -\omega_0^2(\infty) + \alpha_{21}C_1 & -2\beta^2\omega_0(\infty) + 2\alpha_{22}\beta C_2 \end{bmatrix}. \tag{46}$$

In order to guarantee the condition $\lim_{t\to\infty}||B(t)|| = 0$, then from (46) the following conditions must be simultaneously satisfied:

$$\begin{cases} \omega_0^2(\infty) = \alpha_{21}C_1 \\ \omega_0(\infty) = \alpha_{22}C_2, \end{cases} \tag{47}$$

which in turn is equivalent to:

$$C_1 = \frac{\alpha_{22}^2}{\alpha_{21}}C_2^2 \tag{48}$$

3) Stability Remarks: Considering the two conditions analyzed above from the Poincaré-Liapunov theorem in case of first-and second-order systems, the restriction is applied to $\omega_0(\infty)$. This value is defined by:

$$\omega_0(\infty) = -\frac{\lambda_2}{\lambda_1}, \tag{49}$$

considering the first term in (33) and (34) tends to zero when t tends to ∞.

In case of first-order systems, the value of $\omega_0(\infty)$ must fulfill the relations in (39) and (49). While for second-order systems the value of $\omega_0(\infty)$ is also limited by the relation in (47). Finally, it can be concluded that the proposed first-and second-order systems represented by (25) and (26), with the characteristic frequency provided by relations (33) and (34) are bounded-input bounded-output stable [28] given that the value of $\omega_0(\infty)$ is bounded.

B. Time Frequency Description of the Proposed Systems

From the analysis of filter stability the optimal condition for the characteristic frequency function has been derived in case of first-and second-order systems. This is given by the relations in (33) and (34), respectively. These solutions describes not only a bounded function, but also a bounded direct component of $\overline{\omega}_0(t)$.

Let's $W(t,\omega_0)$ be defined by the left side members of relations (33) and (34) as:

$$W(t,\omega_0) = f_i(t,\omega_0) + \lambda_1\omega_0 + \lambda_2, \tag{50}$$

where $i = 1$ determines first-order systems, and $i = 2$ determines second-order systems, respectively. In this case,

$$\begin{aligned} f_1(t,\omega_0) =& t^2\, e^{-t\omega_0}\, (1 - t\omega_0), \\ f_2(t,\omega_0) =& \frac{t^2 e^{-\beta\omega_0 t}}{\sqrt{1-\beta^2}} R\cos(\theta + \phi), \end{aligned} \tag{51}$$

where R, θ and ϕ are given in (17). Then, values of $\overline{\omega}_0(t)$ are obtained from the condition $W(t,\omega_0) = 0$ for each different value of t.

Based on the relations in (50) and (51) it is verified that $W(0,\omega) = W(\infty,\omega) = \lambda_1\omega + \lambda_2$. Thus, the starting and ending values of $\overline{\omega}_0(t)$ are equals and are obtained after equating the above condition to zero as:

$$\overline{\omega}_0(0) = \overline{\omega}_0(\infty) = -\frac{\lambda_2}{\lambda_1}. \tag{52}$$

Values of $\overline{\omega}_0(t)$ are obtained by numerical methods using the condition $W(t,\omega_0) = 0$. The ratio $-\frac{\lambda_2}{\lambda_1}$ establishes the value around which the solution of $\overline{\omega}_0(t)$ is devised.

For instance, taking into account the second order term given by $H_{2i}(s)$ in (9), the characteristic frequency is obtained as depicted in Fig. 2 for a variety of values of λ_1. The plot of $\overline{\omega}_0(t)$ is obtained taking into account the term $f_2(t,\omega_0)$ in (50) when $\overline{\omega}_0(0) = 1000$ and β is fixed to 0.866, similar to lowpass Bessel filters [18]. These curves are encountered by a combination of bisection, secant, and inverse quadratic interpolation methods when t is varied from 0 to $6 \cdot 10^{-4}$ in steps of 10^{-6} seconds.

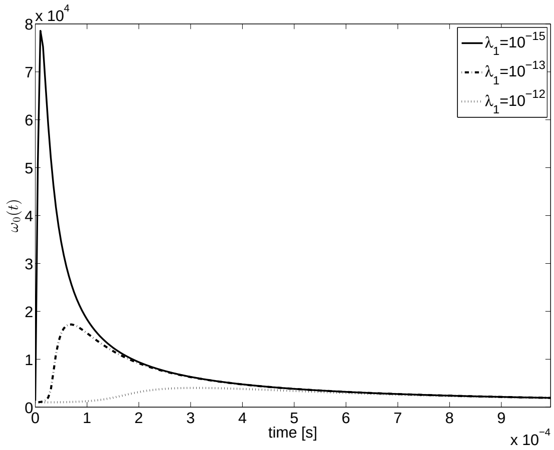

Fig. 2. Characteristic frequency of second order systems.

From Fig. 2, the increment of λ_1 tends to reduce the peak of $\overline{\omega}_0(t)$. This parameter is used to define the range of changes of $\overline{\omega}_0(t)$. The value of λ_2 is selected from relation (49). On the other hand, the variation of $\overline{\omega}_0(t)$ is mostly concentrated near to $t = 0$, where the transient behavior is expected to occur. This represents a common practice to varying the filter bandwidth [22]. Additionally, the behavior depicted on Fig. 2 is similar when terms $f_1(t, \omega_0)$ and $f_2(t, \omega_0)$ are analyzed.

A time frequency description of both proposed systems is provided by the system function of variable networks given by $H(j\omega, t)$ [36]. This function must satisfy the following differential equation:

$$\sum_{n=0}^{N} a_n(t) \frac{\mathrm{d}^n H(j\omega, t)}{\mathrm{d}t^n} = K(j\omega, t), \tag{53}$$

where $a_n(t) = \frac{1}{n!} \frac{\partial^n L(j\omega,t)}{\partial (j\omega)^n}$. Functions $K(j\omega, t)$ and $L(j\omega, t)$ are obtained by the numerator and denominator of the frozen transfer function upon substituting s by $j\omega$. The frozen transfer function is given by the relations in (9) for first-and second-order systems, respectively.

In case of first order system, functions L_1 and K_1 are given by:

$$\begin{aligned} L_1(j\omega, t) =& \frac{j\omega}{\overline{\omega}_0(t)} + 1, \\ K_1(j\omega, t) =& 1, \end{aligned} \tag{54}$$

whereas, in case of second order systems these functions are given by:

$$\begin{aligned} L_2(j\omega, t) =& \frac{1}{\overline{\omega}_0^2(t)} (j\omega)^2 + \frac{2\beta}{\overline{\omega}_0(t)} (j\omega) + 1, \\ K_2(j\omega, t) =& 1. \end{aligned} \tag{55}$$

The differential equation in (52) is solved for both systems by using the frozen system to evaluate the initial and boundary conditions as:

$$\begin{aligned} H(j\omega, 0) =& \frac{K(j\omega, 0)}{L(j\omega, 0)}, \\ H(0, t) =& \frac{K(0, t)}{L(0, t)}, \\ H(j\omega_f, t) =& \frac{K(j\omega_f, t)}{L(j\omega_f, t)}, \\ \frac{\mathrm{d}H(0, t)}{\mathrm{d}t} =& \frac{\mathrm{d}}{\mathrm{d}t} \frac{K(0, t)}{L(0, t)}, \\ \frac{\mathrm{d}H(\omega_f, t)}{\mathrm{d}t} =& \frac{\mathrm{d}}{\mathrm{d}t} \frac{K(j\omega_f, t)}{L(j\omega_f, t)}. \end{aligned} \tag{56}$$

The result is shown in Fig. 3 considering the second order term $H_{2i}(s)$ in (9). A variety of curves for $|H(j\omega, t)|$ are depicted for a variety of time values. These graphs are obtained considering $\beta = 0.866$. Additionally, the characteristic frequency $\overline{\omega}_0(t)$ is selected from $\lambda_1 = 10^{-15}$ in Fig. 2.

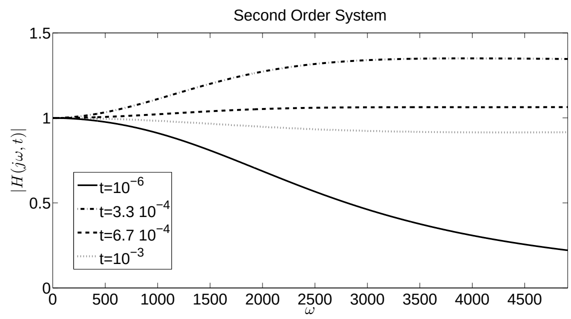

Fig. 3. System function of the proposed second order system.

Curves in Fig. 3 represent a system which tends to increase the allowed passband at the first time interval, then to reduce the passband. This is in accordance with the dynamic established by the corresponding characteristic frequency.

IV. Filter Synthesis

The current section is devoted to describing the implementation of the proposed system. Additionally, the design procedure and an illustrative example sections describe how to implement lowpass filters of arbitrary order.

A. Filter Implementation

The filter is implemented by two connected schemes as depicted in Figure 4, similar to that described in [17]. These schemes are given by the forming scheme to determine the characteristic frequency $\overline{\omega}_{0i}(t)$ and the Filter block. The proper filtering process is implemented by using the obtained expressions for $\overline{\omega}_{0i}(t)$. Additionally, the sequence $\overline{\omega}_{0i}(t)$ is also synchronized with step changes in the incoming signal. To this end, the forming scheme detects the step changes on the incoming signal $x(t)$, then restarts the filtering operations by restarting internal accumulation processes as well as the sequence $\overline{\omega}_{0i}(t)$.

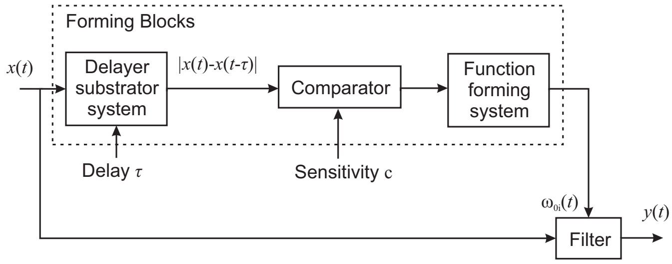

Fig. 4. Block diagrams of LTV filter.

The Filter block is implemented in a parallel form structure comprised by first-and second-order systems as indicated by the relations in (8) and (9). First-and second-order systems are implemented by a state-variable diagram of the differential equations given in (25) and (26) as depicted on Figure 5 [37].

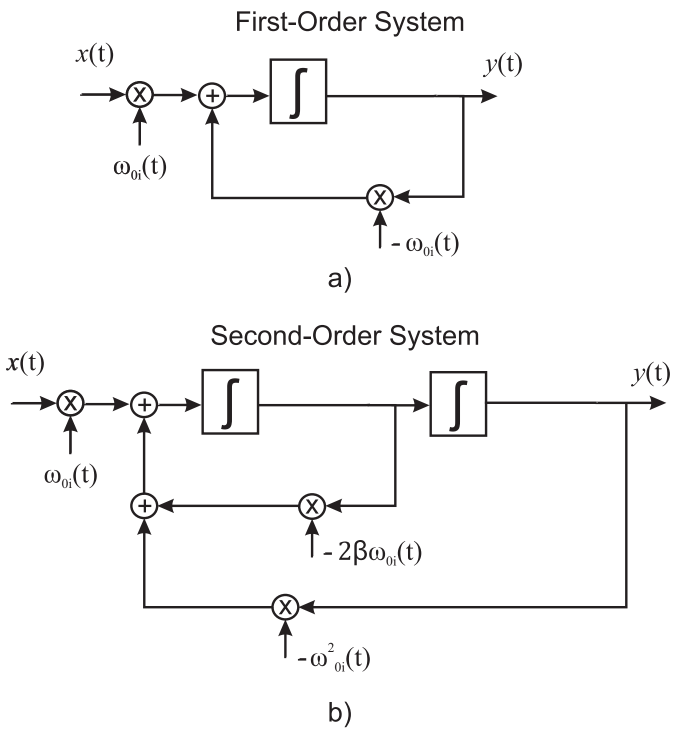

Fig. 5. State-variable diagram for first-and second-order systems. a) First-order system. b) Second-order system.

B. Design Procedure

The starting point of the design procedure for lowpass IIR LTV filters is based on the typical designs of IIR LTI filters. Given an IIR LTI filter, the LTV approach is obtained after incorporating the characteristic frequency derived above for first-and second-order systems. The steps are given as follows:

1) Specify the prescribed parameters to lowpass filtering, such as cutoff frequency, transition band and attenuation in the stop band.
2) Obtain the transfer function by applying approximation methods of the frequency characteristic such as Butterworth, Chebyshev, Bessel, Legendre or Elliptic approximations. Then, decompose the obtained transfer function in terms of first-and second-order systems, as indicated in (8).
3) After establishing a value for λ_1, then for each different term in (9), establish a value of λ_2 by defining $\overline{\omega}_0(0) = \overline{\omega}_0(\infty) = \omega_{0i}$ in (49). The value of ω_{0i} is obtained from each different term in (9). The value of λ_1 is empirically determined and is related with the variation range of the characteristic function $\overline{\omega}_0(t)$.
4) Solve the equations for the characteristic frequency as indicated in (50) by using the parameters λ_1 and λ_2 obtained for each different term in (9). Each different term in (9) will define an specific solution for the characteristic frequency $\overline{\omega}_{0i}(t)$.
5) Implement the filtering operation by a cascade form structure comprised by first-and second-order systems as depicted in Figure 5.

Steps 1 and 2 are related with the invariant filter design. Steps 3 and 4 are related with computing the dynamic parameter of the filter, whereas Step 5 implements the LTV filter. In addition to the five steps above, an allpass system might be connected in cascade to compensate for the resultant phase. Similar to the solutions described in [20]–[22], coefficients of first and second-order allpass systems are determined by using Taylor series expansion or trial and error methods [38] to implement a system of constant group delay. Then, the obtained transfer function is transformed into an LTV filter by steps 2 to 5 above. A detailed discussion is out of the scope of the current paper and further examination is needed.

C. Illustrative Example

The illustrative example is based on Bessel filters, provided it fulfills the best properties in comparison with all filters when rectangular pulses are processed [17]. A variety of systems detect and demodulate signals based on rectangular pulses. This is the case of the envelope detector when an energy detector is employed for device communication with applications in the Internet of Things [39], for instance.

Following the steps from Section IV-B Design Procedure, the LTV filter is obtained as follows:

1) Steps 1 and 2: Taking into account a 4th order Bessel filter of constant parameters, the transfer function is given by [18]:

$$H(s) = \frac{1}{\left(\frac{s^2}{\omega_{01}^2\omega_c^2} + \frac{2\beta_1 s}{\omega_{01}\omega_c} + 1\right)\left(\frac{s^2}{\omega_{02}^2\omega_c^2} + \frac{2\beta_2 s}{\omega_{02}\omega_c} + 1\right)}, \quad (57)$$

where $\omega_{01} = 1.430$, $\beta_{01} = 0.958$, $\omega_{02} = 1.603$ and $\beta_{02} = 0.621$. The constant ω_c stands for the cutoff frequency, in this example $\omega_c = 1000$.

2) Steps 3 and 4: The transfer function provided in (57) is then divided into two second-order terms. Each term defines values of β_1, β_2, ω_{01} and ω_{02}. The value of λ_1 is fixed to 10^{-15}, and the value of λ_2 is determined by (52) in which $\overline{\omega}_1(0) = \omega_{01}\omega_c$ and $\overline{\omega}_2(0) = \omega_{02}\omega_c$. The expressions for $\overline{\omega}_i(t)$ are determined by solving the equation $W(t, \omega_0) = 0$ in (50) by using $f_2(t, \omega)$ to obtain the sequences $\overline{\omega}_1(t)$ and $\overline{\omega}_2(t)$.

3) Step 5: The filter is implemented by discrete techniques connecting in parallel second-order systems with the proper parameters β and $\overline{\omega}_i(t)$ as depicted in Fig. 6. In this case the filter is implemented by using the MatLab R13-Simulink package as shown in Fig. 6 using the structure presented in Fig. 5.

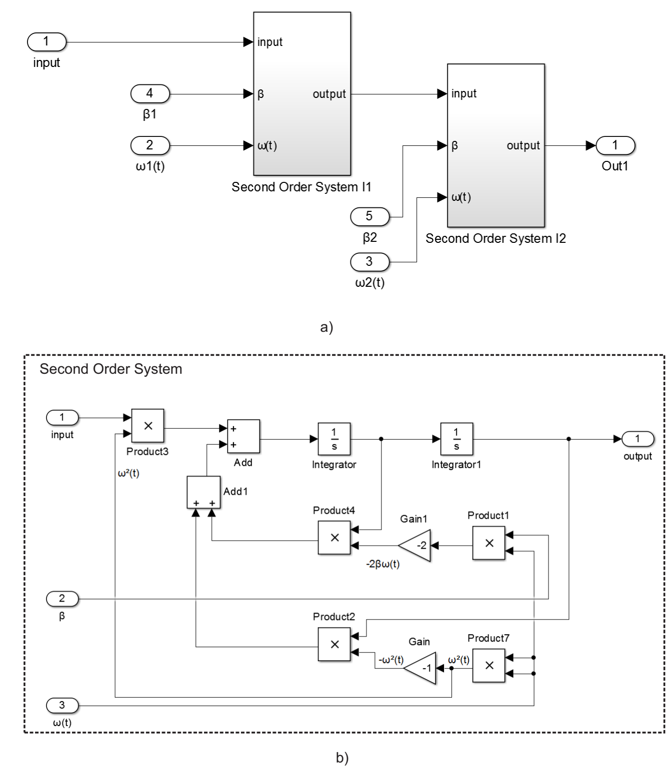

Fig. 6. Block diagram of 4th order LTV filter. a) Cascade connection. b) Second-order filter.

V. Experimental Results

In order to illustrate the performance of the proposed design, graphical results are obtained for the step response in noisy environments. In this case, a fourth-order system is employed as described in [17]. As an example, the signal-to-noise ratio (SNR) parameter is set to 5 dB. The value of $\overline{\omega}_0(0)$ and $\overline{\omega}_0(\infty)$ are set to 1000. Additionally, power and hardware complexity are presented for a circuit level design taking into account the system described in Fig. 6 from the Illustrative Example Section.

For comparing the output filtered signals, three different systems are analyzed: traditional Bessel filter of constant parameter [40], time-varying parameters filter presented by Kaszynski and Piskorowski [17], and the proposed method based on calculus of variations. The fourth-order Bessel filter of constant parameters is implemented as described in (57). On the other hand, the Kaszynski and Piskorowski time-varying filter and the proposed solution, are both implemented similarly, except for the characteristic frequency to be used.

Fig. 7 shows the behavior of the systems analyzed taking into account the rise-time. The proposed solution, the Kaszynski-Piskorowski method and the constant parameter filter exhibit values of T_R equal to $4 \cdot 10^{-6}$, $8 \cdot 10^{-6}$ and $2.2 \cdot 10^{-5}$ seconds, respectively. The proposed solution improves the T_R parameter in comparison to the Kaszynski-Piskorowski method by the half.

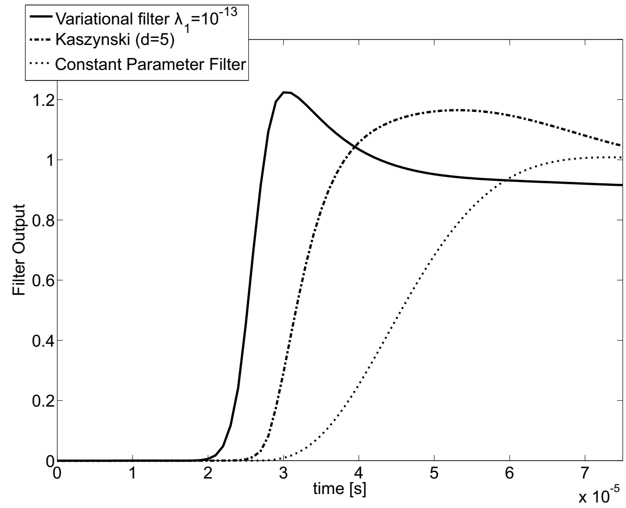

Fig. 7. Output filtered signals by three different systems.

On the other hand, the second simulation is performed by using the input signal with step changes and $SNR = 10$ dB, as depicted on Fig. 8 a). This allows to evaluate the case in which rectangular impulses with certain levels are introduced in the proposed system. The result of filtering is illustrated on Fig. 8 b). The function of the characteristic frequency must be synchronized with the step changes of the input signal by the system depicted on Fig. 4.

From Fig. 8 it can be observed that the proposed filter is superior to the Kaszynski-Piskorowski method in the preservation of the pulse shape. In this regard, the rise-time parameter is reduced on the proposed method in comparison with other approaches for each received pulse.

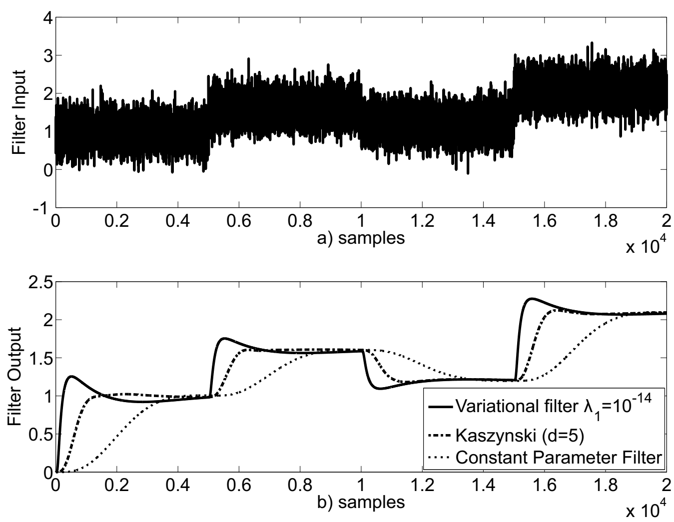

Fig. 8. Filtered signal with step changes.

Finally, considering the system in Fig. 6 a total number of 14 multipliers, 4 adders and 4 integrators are required to implement a 4th order system. The system might be composed of available analog circuits for multipliers [41], adders [42] and integrators [43]. On the other hand, total power consumption driven by these devices is given by 452 mW approximately, taking into account the total amount of multipliers, adders and integrators on the current design. In comparison with other reported solutions on LTV filters, complexity and power consumption are equivalent, provided that system on Fig. 6 is commonly used on these designs. Nevertheless, our proposal minimizes the rise-time parameter while maintaining similar complexity and power requirements, so it outperforms previous LTV filters.

VI. Conclusions and Future Work

In this paper a new filter design approach with time-varying parameters is proposed, which describes an optimal solution for the characteristic frequency by means of calculus of variations. In order to minimize the rise-time parameter, an ideal method and a practical approximation are presented. The implemented system exhibits a reduction in the rise-time parameter with regard to other reported methods. This approach allows overshooting and undershooting due to the variation in the spectral properties. Although, in this respect, the trade-off between rise-time and overshooting is an open issue in the frame work of calculus of variations and time-varying parameters. Additionally, an extension to non-frequency-selective filters is not straightforward and it is out of the scope of this paper. Future work will be focused on several directions: extension to FIR filters, phase compensation for IIR filters, higher order filter implementation based on a parallel connection of first and second order systems, optimum rise-time and overshooting reduction, as well the implementation of the proposed system by discrete-time techniques.

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